Janossy densities of the thinned Airy DPP : Riemann–Hilbert approach

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The Airy determinantal point process is defined as the unique DPP on $\mathbb{R}$ associated to the integral operator $\mathscr{K}^{\mathrm{Ai}}$ on $L^2(\mathbb{R})$. This operator acts through the Airy kernel

$$K^{\mathrm{Ai}}(x,y)=\frac{\mathrm{Ai}(x+s)\mathrm{Ai}'(y+s)-\mathrm{Ai}'(x+s)\mathrm{Ai}(y+s)}{x-y}=\int_0^{+\infty}\mathrm{Ai}(x+t+s)\mathrm{Ai}(y+t+s)dt,$$

where $\mathrm{Ai}(\cdot)$ stands for the classical Airy function, i.e. a rapidly decaying at $+\infty$ real solution of the Airy equation $f''(x)=xf(x)$.

Hermitian locally trace class integral operator $\mathscr{K}$ on $L^2(\mathbb{R})$ defines a determinantal point process on $\mathbb{R}$ if and only if $0\le\mathscr{K}\le 1$. If the corresponding point process exists it is unique. [4]

The gap probability: $\sigma(r)=\chi_{(s,\infty)}(r)$ vs $(1+e^{-r\alpha})^{-1}$

The probability distribution function of the largest particle of the Airy DPP is described by the Fredholm determinant $F(s)=\det(1-\mathscr{K}^{\mathrm{Ai}}|_{(s,+\infty)})$.

$F(s)$ is described by the so called *Tracy–Widom* formula

$$\frac{d^2}{ds^2}\ln F(s)=-u^2(s)$$

where u is the Hastings-McLeod solution of the *Painlevé II* equation, i.e. it solves the equation $u''(s)=su(s)+2u^3(s)$ together with the boundary condition $u(s)\sim\mathrm{Ai}(s)$ for $s\to+\infty$. [5]

Let $\sigma:\mathbb{R}\to[0,1]$ be a non-decreasing smooth function (e.g. the Fermi factor) and consider the Fredholm determinant $F_\sigma(s)=\det(1-\sigma\mathscr{K}_s^{\mathrm{Ai}})$ where now the integral operator $\mathscr{K}_s^{\mathrm{Ai}}$ has kernel $K_s^{\mathrm{Ai}}(x,y)=K^{\mathrm{Ai}}(x+s,y+s)$.

$F_\sigma(s)$ satisfies the analogue of *Tracy–Widom* formula

$$\frac{d^2}{ds^2}\ln F_\sigma(s)=-\int_{\mathbb{R}}\varphi^2(\lambda;s)\sigma'(\lambda)d\lambda$$

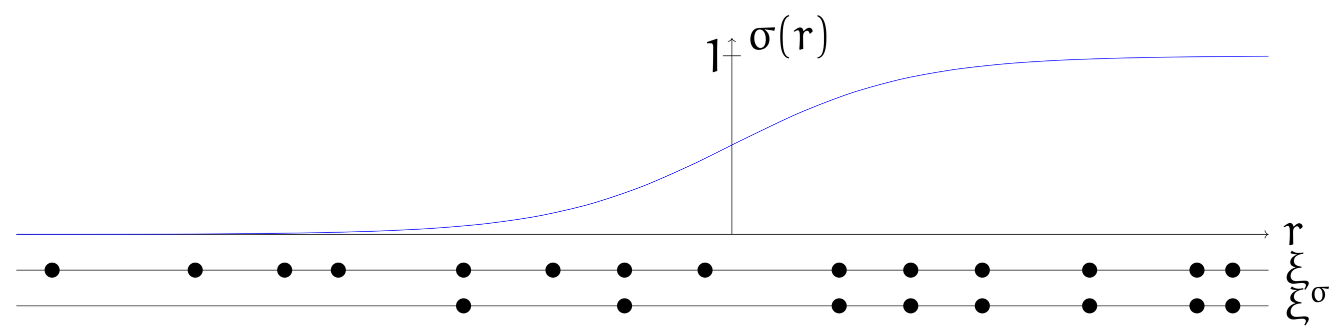
where this time $\varphi(\lambda;s)$ satisfies the *integro-differential Painlevé II* equation

$$\frac{\partial^2}{\partial s^2}\varphi(\lambda;s)=\left(\lambda+s+2\int_{\mathbb{R}}\varphi^2(r;s)\sigma'(r)dr\right)\varphi(\lambda;s)$$

and the boundary condition $\varphi(\lambda;s)\sim\mathrm{Ai}(\lambda+s)$ for $s\to+\infty$, pointwise in λ. [1]

Thinning operation

Let $\mathscr{K}$ be an integral operator defining a DPP $\mathbb{P}$ on $\mathbb{R}$ and consider a function σ as before. Then $\sigma\mathscr{K}$ gives the σ-*thinned* process $\mathbb{P}^\sigma$, that is constructed from every random configuration ξ in $\mathbb{P}$ by independently removing a particle ξ_j in the configuration ξ with probability $1-\sigma(\xi_j)$ and by keeping it with probability $\sigma(\xi_j)$.



The Janossy densities

Let $m\in\mathbb{N}$ and $V=\{v_1,\dots,v_m\}\subset\mathbb{R}$. If $\sigma\mathscr{K}_s^{\mathrm{Ai}}$ is trace-class then $\mathbb{P}^{\mathrm{Ai}_s;\sigma}$ has a. s. # particles $<\infty$ and we can then define Janossy densities of the thinned shifted Airy DPP

$$J_\sigma(V|s)=\det(1-\sigma\mathscr{K}_s^{\mathrm{Ai}})\det_{1\le k,h\le m}(L_{\sigma,s}^{\mathrm{Ai}}(v_k,v_h))=F_\sigma(s)\rho_\emptyset^{\sigma,s}(v_1,\dots,v_m)$$

where $L_{\sigma,s}^{\mathrm{Ai}}$ is the kernel of the operator $\mathscr{L}_{\sigma,s}^{\mathrm{Ai}}$ defined as

$$\mathscr{L}_{\sigma,s}^{\mathrm{Ai}}=\mathscr{K}_s^{\mathrm{Ai}}\left(1-\sigma\mathscr{K}_s^{\mathrm{Ai}}\right)^{-1}.$$

Remark: $\rho_\emptyset^{\sigma,s}(v_1,\dots,v_m)$ is actually the correlation function of another DPP: obtained by first σ—marking with 0 and 1 the shifted Airy DPP and then conditionning on empty 1 configurations the marked process. [2]

Theorem #1

Also $J_\sigma(V|s)$ is characterized in terms of the eigenfunction $\varphi(\lambda;s,\emptyset)=\varphi(\lambda;s)$, since

$$L_s^\sigma(\lambda,\mu)=\int_s^{+\infty}\varphi(\lambda;s')\varphi(\mu;s')\,ds'=\frac{\varphi(\lambda;s)\varphi'(\mu;s)-\varphi'(\lambda;s)\varphi(\mu;s)}{\lambda-\mu},$$

and

$$F_\sigma(s)=\exp\left(-\int_s^{+\infty}(s'-s)\left(\int_{\mathbb{R}}\varphi(\lambda;s')^2d\sigma(\lambda)\right)ds'\right).$$

The function $\varphi(\lambda;s,\emptyset)$ solves the Schrödinger equation

$$\left[\partial_s^2+2u(s,\emptyset)\right]\varphi(\lambda;s)=\lambda\varphi(\lambda;s),\ \text{with potential}\ u(s,\emptyset)=-\int_{\mathbb{R}}\varphi^2(r,s)\sigma'(r)dr-\frac{s}{2}.$$

Theorem #2

We also have a *Tracy–Widom* characterization for $J_\sigma(V|s)$ in the sense that

$$\frac{d^2}{ds^2}\ln J_\sigma(V;s)=-\int_{\mathbb{R}}\varphi^2(r;s,V)\,\sigma'(r)dr\,-\,2\pi\sum_{v\in V}\lim_{r\to v}\varphi(r;s,V)\widetilde{\varphi}(r;s,V).$$

Here $\varphi(z;s,V),\widetilde{\varphi}(z;s,V)$ solve both the Schrödinger equation

$$\left[\partial_s^2+2u(s,V)\right]\begin{pmatrix}\varphi(\lambda;s,V)\\ \widetilde{\varphi}(\lambda;s,V)\end{pmatrix}=z\begin{pmatrix}\varphi(\lambda;s,V)\\ \widetilde{\varphi}(\lambda;s,V)\end{pmatrix},$$

with $u(s,V)=-\int_{\mathbb{R}}\varphi^2(r;s,V)\,\sigma'(r)dr\,-\,2\pi\sum_{v\in V}\lim_{r\to v}\varphi(r;s,V)\widetilde{\varphi}(r;s,V)-\frac{s}{2}$.
Notice that $\pi\lim_{r\to v_j}\varphi(r;s,V)\widetilde{\varphi}(r;s,V)=-\sum_i^m\mathbb{L}_{j,i}^{-1}(s|V)\varphi(v_i;s,\emptyset)\varphi'(v_j;s,V)$.

The Riemann–Hilbert problem

(a) $\Psi(\cdot;s|V):\mathbb{C}\setminus\mathbb{R}\to\mathbb{C}^{2\times2}$ is analytic for all $s\in\mathbb{R}$ and all finite $V\subset\mathbb{R}$.

(b) The boundary values of $\Psi(\cdot;s|V)$ are continuous on $\mathbb{R}\setminus V$ and are related by

$$\Psi_+(\lambda;s|V)=\Psi_-(\lambda;s|V)\begin{pmatrix}1&1-\sigma(\lambda)\\0&1\end{pmatrix},\qquad\lambda\in\mathbb{R}\setminus V.$$

(c) For all $v\in V$, as $\lambda\to v$ from either side of the real axis we have

$$\Psi(\lambda;s|V)(\lambda-v)^{-\sigma_3}=O(1).$$

(d) As $\lambda\to\infty$, we have $\Psi(\lambda;s|V)=\left(I+O(\lambda^{-1})\right)\lambda^{\sigma_3/4}A^{-1}e^{\left(-\frac{2}{3}\lambda^{\frac{3}{2}}-s\lambda^{1/2}\right)\sigma_3}C_\delta$ for any $\delta\in(0,\frac{\pi}{2})$. Here we take the principal branches of $\lambda^{\sigma_3/4}$ and $\lambda^{1/2}$, analytic in $\mathbb{C}\setminus(-\infty,0]$ and positive for $\lambda>0$, and

$$\sigma_3:=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\qquad A:=\frac{1}{\sqrt{2}}\begin{pmatrix}1&i\\i&1\end{pmatrix},\qquad C_\delta:=\begin{cases}I, & |\arg\lambda|<\pi-\delta,\\ \begin{pmatrix}1&0\\\mp1&1\end{pmatrix}, & \pi-\delta<\pm\arg\lambda<\pi.\end{cases}$$

- Let $Y(\lambda;s)$ be the solution of the RH problem associated to the integrable IIKS [3] kernel $\sigma\mathscr{K}_s^{\mathrm{Ai}}$ and $\Phi_{\mathrm{Ai}}(\lambda+s)$ is the (shifted) solution to the model Airy RH problem. Then

$$\Psi(\lambda,s;\emptyset)=\begin{pmatrix}1&\frac{is^2}{4}\\0&1\end{pmatrix}Y(\lambda;s)\Phi_{\mathrm{Ai}}(\lambda+s).$$

- $\Psi(\lambda;s,V)=R(\lambda;s,V)\Psi(\lambda;s,\emptyset)$ where $R(\lambda;s,V)$ is a rational function of λ with poles at $\lambda\in V$ only, given by

$$R(\lambda;s,V)=I-\frac{1}{2\pi i}\sum_{i,j=1}^m\frac{\mathbf{L}_{j,i}^{-1}(s;V)}{\lambda-v_j}\Psi(v_i;s,\emptyset)\begin{pmatrix}0&1\\0&0\end{pmatrix}\Psi^{-1}(v_j;s,\emptyset),$$

where $\mathbf{L}(s,V)$ the square matrix of size m with entries $\mathbf{L}_{k,h}(s,V):=L_{\sigma,s}^{\mathrm{Ai}}(v_k,v_h)$.

- $$L_{\sigma,s}^{\mathrm{Ai}}(v,w;s)=\frac{\left(\Psi^{-1}(w;s,\emptyset)\Psi(v;s,\emptyset)\right)_{2,1}}{2\pi i(v-w)},v\neq w,$$

  and analogously for $v=w$.

- $$\Theta(\lambda;s,V):=\begin{pmatrix}1&p(s,V)\\0&1\end{pmatrix}e^{\frac{i\pi}{4}\sigma_3}\Psi(\lambda;s,V)e^{-\frac{i\pi}{4}\sigma_3}=\sqrt{2\pi}\begin{pmatrix}\partial_s\varphi(\lambda;s,V)&\partial_s\widetilde{\varphi}(\lambda;s,V)\\ \varphi(\lambda;s,V)&\widetilde{\varphi}(\lambda;s,V)\end{pmatrix}.$$

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